A DESCRIPTION OF STUDENT RESPONSES TO RESTRICTIONS ON ASSUMED KNOWLEDGE

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Abstract
This paper investigates indicators of formal thinking in geometry problems. A sample of 20 tertiary students were given a test item which restricted the information that was available to them for use in the proof. The test item was administered before and after a course on the development of non-Euclidean geometry. Before the course, the students were not able to deal adequately with restrictions to the knowledge they could assume. After the course, students appeared to cope well with the notion of restrictions to assumed knowledge, but this accommodation did not translate into the writing of acceptable proofs. The conclusions drawn are: (i) writing proofs where there are restrictions to assumed knowledge clearly involves a sophisticated form of deductive reasoning; and, (1i) an indicator of advanced formal thinking is the ability to develop a proof where cases, which take into account different relationships (as opposed to properties) between necessary and sufficient conditions, are accommodated.

Introduction

Booth (1990), writing about her work in Algebra, described a paradigm shift in the development of mathematical thinking from what she calls "empirical mathematics" to "invented mathematics". The former is related to real objects and events. The latter is concerned with the study of mathematical concepts and mathematical objects for their own sake. In particular, the focus is on logical relationships. Pegg and Coady (1993), also researching in the Algebra field, provided some insight into the more advanced thinking that occurs at the upper boundary of this work in the senior secondary school. They investigated student responses to items that involved solving algebraic inequalities and concluded that the ability to consider various cases which take into account a range of possibilities and accompanying limitations is an indicator of formal thinking.

Geometry provides another example of this paradigm shift. In Australia, the study of geometry most commonly begins empirically through investigations of concrete shapes. The emphasis then moves to proof where deductive reasoning and logical relationships are the focus. The theory of Van Hiele (1986) can be interpreted also within the broad framework described above and provides a deeper insight into the structure of (in this case) geometrical thinking. In this Theory, students' understanding can be interpreted within a scheme of five levels (Levels 1 to 5). The first two involve recognition of geometrical figures by their shape and their properties, respectively. These Levels fall clearly within the empirical paradigm of Booth. Level 3, in which relationships are developed between the properties identified at Level 2 and an ordering of the properties of geometrical figures is possible, represents the 'middle ground'. Here the first instance of deductive argument, albeit in a simple form, can be found. Also at this level, notions of parallelism and congruency are understood. That is, students can state the rules of congruency for triangles and answer questions concerning angles, given parallel lines. However, the ability to understand formal proofs has not yet been reached.

The remaining two levels, Levels 4 and 5, lie completely within the logical paradigm. At Level 4, formal thinking occurs and is evidenced by students being able to construct for themselves geometric proofs that have not been learnt by rote. At this Level concepts such as parallelism and congruency become tools to be used. They are used as elements in their own right. In essence, the nature of deduction is understood.

Students functioning at Level 5 are able to adequately challenge the assumptions and postulates that underpin the structures at Level 4. As a result, much of the content that encapsulates Level 5 thinking can be considered under the broad rubric of non-Euclidean geometry. To provide a basis for this study a brief survey is provided of the relevant mathematics.

Background

In Euclidean geometry, the fifth and final postulate (axiom) which is also called the parallel postulate states:

> *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.*

This postulate is much more complicated that the previous four. There is a long history of famous mathematicians who thought it 'unpleasant' and tried to prove that it was not necessary. They provided 'proofs' that the Parallel Postulate was a theorem which followed logically from the first four postulates. Unfortunately, all such 'proofs' had some other hidden assumption which was equivalent to the fifth postulate. One famous example is that of John Playfair (1748 - 1819) who showed the fifth postulate was not necessary by implicitly assuming that:

> *Through a given point not on a given line, only one straight line parallel to the given line can be drawn.*

The equivalence of Playfair's Axiom and the Parallel Postulate is a standard result in modern geometry.

Lobachevsky (1830) and Boylai (1832) showed that it was possible to have a geometry where a point not on a given line, could have more than one straight line parallel to the given line drawn through it. So entered non-Euclidean geometry and the realisation that results such as alternate angles are equal for parallel lines, the angle sum of a triangle is 180°, and Pythagoras' Theorem are not absolute, but depend on the Parallel Postulate. It is clear, given van Hiele's description, that what has been discussed above are examples of Level 5 thinking.

However, the history of 'proofs' of the Parallel Postulate provides many simpler examples of mathematicians restricting assumed knowledge to the first four postulates of Euclid. One famous example is the 'Saccheri Quadrilateral' pictured in Figure 1. This figure has opposite sides equal and base angles right angles.

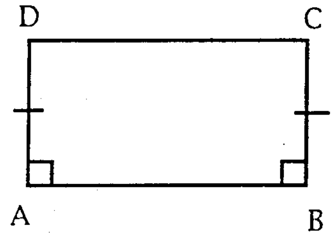

Figure 1

The purpose of the example was to prove that $\angle C = \angle D$ in Figure 1 with the assumed knowledge restricted to the first four postulates of Euclid. This restriction meant that no use of any result that followed from the Parallel Postulate or equivalent was permitted. Using any such result was deemed as using 'forbidden knowledge'. The standard proof (see for example, Lockwood & Runion, 1978: 18) that $\angle C = \angle D$ is in two parts and makes use of congruent triangles.

Part (i): In Figure 2, AB common yields Δ CAB $\cong$ Δ DAB (SAS). Hence, CB = AD.

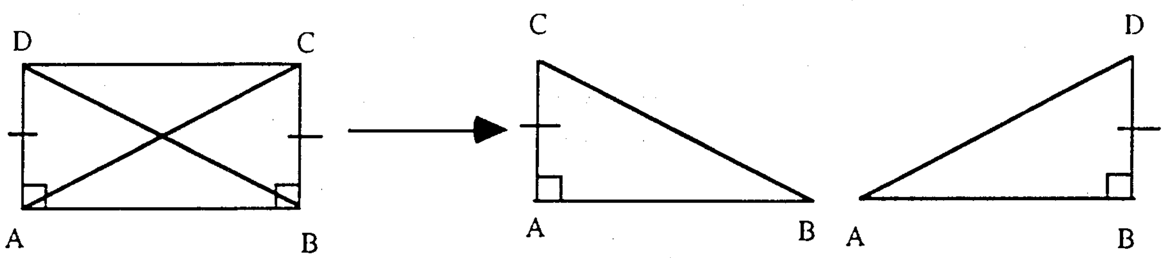

Figure 2

Part (ii): In Figure 3, CB = AD and CD is common
Hence, Δ ACD $\cong$ Δ BDC (SSS) and $\angle C = \angle D$.

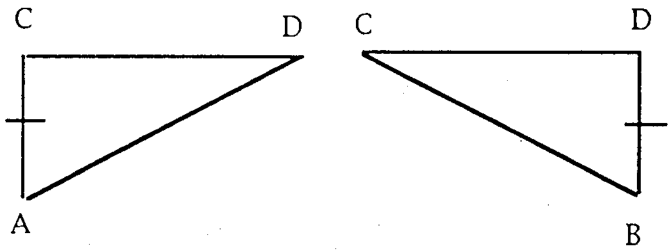

Figure 3

This proof requires Level 4 thinking. Students need to: use congruency as a tool; be able to work with constraints to assume knowledge; and, at the same time, be able to undertake a deductive process.

Design

The Saccheri Quadrilateral question was the last of five test items administered to 20 students studying the development of non-Euclidean geometry, both before (pre-test) and after (post-test) the study of the Parallel Postulate. All students had previously passed a course in Euclidean geometry where an acceptable proof could use data from a large pool of unstructured, general results without thought given to a possible ordering or hierarchy of the assumed knowledge. During the study of the Parallel Postulate, students worked with proofs that required a restriction on assumed knowledge, but not specifically with the Saccheri Quadrilateral. To assist students in the pre-test situation some specific results were given with the item as forbidden knowledge. These were: the angle properties of parallel lines; the angle sum of a triangle; and, Pythagoras' Theorem. Hence, for example, students were not able, to deduce the answer by showing the figure to be a rectangle. All students were interviewed after each data collection to clarify the reasons for their written responses and to gauge their reactions at being confronted with restrictions to assumed knowledge. The first interview also clarified students' general strategies for writing geometric proofs.

The research questions that focussed the investigation were:

1. How difficult was the notion of forbidden knowledge to students who had already shown ability at deductive thinking?
2. Did the clarification of the context (i.e., a short course of study in non-Euclidean geometry) assist students and if so in what ways?
3. What implications can be drawn from the results which provide insight into advanced Level 4 thinking?

Results

In both data collections, student responses fell into the following five categories:

1. Correct.
2. Incorrect with **no** use of forbidden knowledge.
3. Incorrect with use of forbidden knowledge or false assumptions, but with the realisation that forbidden knowledge or false assumptions had been used.
4. Incorrect with use of forbidden knowledge or false assumptions, but without the realisation that forbidden knowledge or false assumptions had been used.
5. No attempt.

The number of responses in each category for both pre- and post-test.

Category	Pre-Test	Post-Test
1. Correct	0	1
2. No forbidden knowledge	3	4
3. Realised forbidden knowledge	0	4
4. Not realised forbidden knowledge	5	6
5. No attempt	12	5
Total	20	20

Discussion

Pre-Test

The general reaction to the test item by the students was one of surprise. The three students who attempted but did not use forbidden knowledge drew diagonals on the quadrilateral so that triangles were formed. They then gave up because they could not use the angle sum of a triangle or think of other strategies. Interviews showed that the students who did not attempt the question had trouble conceiving the idea of not being able to use standard results. Those who used forbidden knowledge without realisation did not use any of the given direct results of the Parallel Postulate such as: the angle sum of a triangle is 180^o, but rather, deductions following from these results. For example, four of the five students who used forbidden knowledge without realisation, used the property: the angle sum of a rectangle is $360^{o'}$. However, when reflecting on their answer in the interview, three students concluded that they had used forbidden knowledge as the angle sum of a rectangle did depend on the angle sum of a triangle being 180^o.

In summary, the students were overwhelmed when they were confronted with restrictions to assumed knowledge. They were either unable to continue or drew further inadmissible conclusions about rectangles which on reflection they realised were inadmissible. All chosen strategies were limited to the specific context of the item. Equal angles were required and so the students focused their attention only on angles. No one considered the use of congruent triangles, a common strategy for proofs where triangles are involved.

Finding 1

Restricting assumed knowledge challenged the students' basic idea of what constitutes proof in geometry. Hence, proofs with restrictions to assumed knowledge are clearly qualitatively different to proofs where there is ready access to an unstructured pool of knowledge.

Post-Test

The one student who was correct gave the proof shown in Figure 4. It is in two parts.

Part (i): Join D and C to X, the mid-point of AB. Then Δ ADX $\cong$ Δ BCX (SAS).

Hence, $\angle$ ADX = $\angle$BCX and DX = CX,

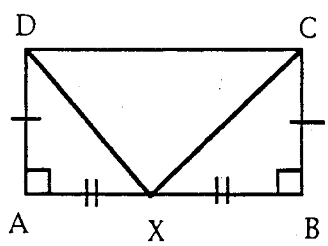

Figure 4

Part (ii): DX = CX makes Δ XDC an isosceles, triangle and $\angle$X DC = $\angle$XCD.

Now, $\angle$C = $\angle$ADX + $\angle$X DC = $\angle$BCX + $\angle$XCD = $\angle$D.

This proof is different from the standard proof referred to earlier and is quite elegant. The reason for this is that the construction is to the mid-point and the second part is much more direct than proving more triangles are congruent. The key to the success of this student was: (i) the ability to select strategies beyond the immediate context of the item; and (ii) the ability to prove one result and then realise that this result was a basis for establishing the desired proof. This can be described as operating in a strategy → result → strategy → result cycle.

Of the four students who used **no** forbidden knowledge, only three made any real progress. One student drew the same construction as shown in Figure 4, completed part (i), deduced that the two base angles of the isosceles triangle were equal, but did not complete part (ii). No link was made between the equal angles in the isosceles triangle and the angles proved equal in part (i). The other two students, who used no forbidden knowledge, showed similar ability to the student just described. They used the construction of the standard proof, illustrated in Figure 2, and completed part (i) correctly but did not continue with the construction used in Figure 3.

Finding 2

Selecting permitted strategies beyond the immediate context of the item indicates an ability to deal with the imposed restrictions. Operating in a strategy → result → strategy → result cycle indicates a higher level of operation where not only the restrictions to assumed knowledge are accommodated, but also links are formed between the relationship of necessary and sufficient conditions in one strategy → result action with the next.

The four students who knew they had used forbidden knowledge admitted they had done so "because it was better than writing nothing". The five students who made no attempt were similar to these four because they too could see no strategy that was permissible but chose to write nothing. Of the six students who used forbidden knowledge or false assumptions without realisation, two of the six used no forbidden knowledge, but made the unfounded assumption that the diagonals bisected the angles at the vertices. Two more used the result about the angle sum of a rectangle being 360°. The remaining two students correctly proved AB was parallel to CD, but then used properties of a parallelogram that depend on the Parallel Postulate - a far more subtle use of the forbidden properties of the rectangle than used by the other two students.

In summary, an increased number of students, on the post-test, have chosen strategies beyond the direct consideration of angles only. Ten students introduced congruent triangles in some form and two students correctly proved AB was parallel to CD. This was a major, if limited, improvement on the pre-test attempts. However, most of these strategies resulted in little real progress towards a correct proof and can be easily attributed to a heighten awareness caused by the teaching programme. Students were now aware of **exactly** what results were forbidden and what results were permissible. One indication of this awareness occurred when three students wrote a 'shopping list' on their test papers of all the results that they could and could not use.

Interviews showed that all students had a similar list, at least in their minds, when doing the test question.

Finding 3

While understanding about and practise with ideas associated with assumed knowledge does lead students to consider a greater range of approaches, this does not translate automatically into effective strategies for writing a correct proof.

Conclusion

Finding 1 argues that there is a qualitative difference between proofs with restrictions to assumed knowledge and proofs without restrictions. However, this difference is not a clear dichotomy as findings 2 and 3 indicate a hierarchy in the procedures used. This hierarchy can be described within three broad categories, namely: (i) no ability to choose a strategy beyond the immediate context of the question; (ii) an ability to choose a strategy beyond the immediate context of the question, but only cope with one strategy → result action; and, (iii) an ability to choose a strategy beyond the immediate context of the question and cope with a strategy → result → strategy → result cycle.

The students in category (i) avoided using forbidden knowledge and thus appeared to show some degree of formal logical reasoning. However, that they used no forbidden knowledge has been shown to be the result of having a written 'shopping list' of what could and could not be used. The presence of this list and their inability to select a strategy beyond the immediate context of the question raises doubts about the real quality of their formal reasoning. In reality, their ability to cope with restrictions to assumed knowledge was clearly limited.

The obstacle to writing a correct proof for the students in category (ii) was not their inability to cope with restrictions to assumed knowledge, or their inability to select a strategy beyond the immediate context of the item. Instead it was their inability to form links between the relationship of a necessary and sufficient condition in one strategy → result action and the relationship of a necessary and sufficient condition in a second strategy → result action. Hence, the ability to cope with more than one relationship was a significant impediment for these students in formulating their response.

Implications

This study has some links with the work in Algebra of Pegg and Coady (1993). Their research indicated that formal reasoning is required when various cases need to be considered at the same time. The item that asked students to solve the inequality $\frac{1}{p} > p$ required the cases $p < 0$ and $p > 0$ to be considered. Taking each case separately requires restricting useable information about p. The restriction, though, is limited to a property of p (p positive or p negative). The question described in this paper also involves restrictions. However, these apply not just to the properties of triangles and rectangles, but involve relationships between necessary and sufficient conditions. It would seem that the skills required here are of a much higher order than those identified in the Algebra investigation.

This study represents an initial attempt at trying to unravel the ways capable tertiary mathematics students develop understandings of and strategies for more difficult concepts. It is possibly the first time that a study has been specifically directed about the interface of van Hiele's Level 4 and Level 5. As a result the investigation was exploratory and the findings and conclusions tentative. However, further research directions are clear. It is important to clarify the roles of restrictions to assumed knowledge as well as the linking of more than one relationship in a strategy → result → strategy → result cycle. Items are needed that can differentiate between: (i) no restrictions to assumed knowledge with one strategy → result action and no restrictions to assumed knowledge with two strategy → result actions; (ii) no restrictions to assumed knowledge with one strategy → result action and restrictions to assumed knowledge with one strategy → result action; and, (iii) restrictions to assumed knowledge with one strategy → result action and restrictions to assumed knowledge with two strategy → result actions.

In terms of Level 4 thinking, this paper offers the hypothesis that there is a hierarchy of growth in understanding Geometry. This can be described in broad terms as: leading from being able to consider various cases that involve the interaction of different properties of mathematical objects to the consideration of various cases in which the interaction involves different relationships. Further, it is this last feature that accounts for most of the jump in cognitive load experienced by students when dealing with questions at this Level.

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